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Faces of Platonic solids in all dimensions

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This paper considers Platonic solids/polytopes in the real Euclidean space $\mathbb{R}^n$ of dimension $3 \le n < \infty$. The Platonic solids/polytopes are described together with their faces of dimensions $0 \le d \le n-1$. Dual pairs of Platonic polytopes are considered in parallel. The underlying finite Coxeter groups are those of simple Lie algebras of types A_n, B_n, C_n, F_4, also called the Weyl groups or, equivalently, crystallographic Coxeter groups, and of non-crystallographic Coxeter groups H_3, H_4. The method consists of recursively decorating the appropriate Coxeter–Dynkin diagram. Each recursion step provides the essential information about faces of a specific dimension. If, at each recursion step, all of the faces are in the same Coxeter group orbit, *i.e.* are identical, the solid is called Platonic. The main result of the paper is found in Theorem 2.1 and Propositions 3.1 and 3.2.

1. Introduction

Platonic solids are understood here as the subset of polytopes whose vertices are generated, starting from a single point in $\mathbb{R}^n$, $n \ge 3$, by the action of a finite Coxeter group. Platonic polytopes are distinguished by the fact that their faces f_d of any dimension $0 \le d \le n-1$ are Platonic solids of lower dimension, and that they are transformed into each other by the action of the Coxeter group, *i.e.* they belong to one orbit of the corresponding Coxeter group.

It has been known since antiquity that there are five Platonic solids in $\mathbb{R}^3$ (Coxeter, 1973), namely, the regular tetrahedron, cube, octahedron, icosahedron and dodecahedron (see Fig. 1). The underlying Coxeter groups[1] are A_3, B_3, C_3 and H_3. They are the lowest-dimensional cases which serve here as transparent examples for our method.

Our method consists of recursive decorations of the nodes of corresponding Coxeter–Dynkin diagrams. The method was developed in Moody & Patera (1992) and Champagne *et al.* (1995); see also the proceedings Moody & Patera (1993). Here it is used for the first time for the description of faces of Platonic solids in the real Euclidean space $\mathbb{R}^n$ of dimension n. Each decoration provides a description of a face representing the conjugacy class of the faces of the polytope. Nodes of the diagram stand for reflections r_k generating the corresponding Weyl or Coxeter group. By r_k we denote the reflection in the $(n-1)$-dimensional hyperplane in $\mathbb{R}^n$ orthogonal to the simple root α_k and containing the origin.

The root systems of finite Coxeter groups of any type (Deodhar, 1982) allow our method to be extended to higher dimensions. For $\mathbb{R}^4$, a classification of Platonic solids was done more than a century ago by Schläfli (1855). In this case, the Coxeter groups are A_4, B_4, C_4, F_4 and H_4. It is also known that, in any dimension ≥ 5, there are only three such solids generated by groups of types A_n, B_n and C_n, namely the n-dimensional simplex that is an analog of the tetrahedron, hypercube and cross-polytope. They correspond to the regular tetrahedron, cube and octahedron in $\mathbb{R}^3$.

In this paper, we derive these results by rather simple rules for recursive decoration for any connected Coxeter–Dynkin diagram (Champagne *et al.*, 1995). There are three different decorations of the nodes of a diagram, namely $\circ$, $\bullet$ and $\diamond$. Directly from a decoration, one reads the reflections generating the symmetry group of a face and at the same time the reflections generating the stabilizer of the face in the Coxeter group of the diagram. We also provide a constructive method for building faces f_d of dimensions $0 \le d \le n-1$. Although we focus on polytopes of dimension $d \ge 3$, it is also useful to consider $n = 2$ Platonic solids because they occur as two-dimensional faces of higher-dimensional polytopes. Moreover, the diagram decoration method can be used in much more general situations and, indeed, in spaces with a non-Euclidean metric and even in spaces where the metric is not known (Moody & Patera, 1995).

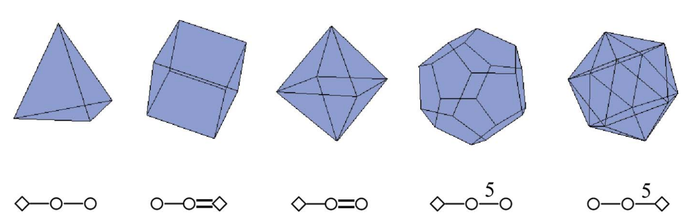


Figure 1
The Platonic solids in the three-dimensional case: tetrahedron, cube, octahedron, dodecahedron, icosahedron. Below each solid there is a seed point for each Coxeter–Dynkin diagram.

[1] Finite reflection groups, called Coxeter groups, are denoted here by symbols commonly used for respective simple Lie algebras (Humphreys, 1990). Finite Coxeter groups with no connection to Lie algebras are denoted by H_2, H_3, H_4.

For every Platonic solid there exists its dual, which is also a Platonic solid. We describe both members of each dual pair. For $n \geq 3$, the dual pair of A_n consists of two identical solids oriented differently in space. The Platonic solids of B_n and C_n form the dual pair. For F_4, H_3 and H_4, the dual pairs are obtained by interchanging the roles of the reflections marked by $\circ$ and $\bullet$. They form different solids.

The general idea of the diagram decoration method (Champagne *et al.*, 1995) is to consider the Coxeter group $W(\mathfrak{g})$ of the simple Lie algebra $\mathfrak{g}$, that is, the symmetry group of a given solid, and to identify the subgroup $G_s(f_d)$ that pointwise stabilizes the given face f_d, and the subgroup $G_f(f_d)$, which is the symmetry group of the face f_d. Then we have

$$G_s(f_d) \times G_f(f_d) \subset W(\mathfrak{g}), \quad 0 \leq d \leq n-1. \tag{1}$$

Since both $G_s(f_d)$ and $G_f(f_d)$ are generated by reflections defined by some of the simple roots of $\mathfrak{g}$, it is possible, indeed convenient, to distinguish by different decoration the simple roots of the Coxeter–Dynkin diagram defining the reflections generating the two subgroups. The decorated diagrams allow one to identify all of the polytope faces that belong to different orbits of the corresponding Coxeter groups, and to count how many times each face occurs on the polytope.

We decorate nodes of the Coxeter–Dynkin diagram by one of the three symbols $\bullet$, $\diamond$ and $\circ$, according to the rules in §2.2.[2]

Decorated Coxeter–Dynkin diagrams are a powerful method of great generality (Moody & Patera, 1995), which could be used to solve other problems, but which so far remain underused. In Moody & Patera (1992), the method was used to describe Voronoi and Delone cells in root lattices of all simple Lie algebras $\mathfrak{g}$. It still has to be used to describe the Voronoi and Delone cells in weight lattices (Conway & Sloane, 1998), and for other problems as well (Fowler & Manolopoulos, 2007; McKenzie *et al.*, 1992).

In this paper, we use a version (Champagne *et al.*, 1995) of the method to describe the Platonic solids together with all of their faces. In dimension 4, equivalent results can be found among the entries of Table 3 of Champagne *et al.* (1995).

Decoration rules are recursive. Starting from a seed decoration, which provides information about the vertices f_0 of the polytope, the procedure consists of modifying the decoration step by step. At each step, the modified decoration provides information about the faces of dimension greater by 1. Decoration rules are quite general and are particularly simple when applied to Platonic polytopes. The same set of decorations applies to Coxeter–Dynkin diagrams with the same number of nodes. As the links between diagram nodes do not affect decoration rules, the links need not be drawn. Only when working with a specific reflection group, the decorated diagram with all links removed is viewed superimposed on the appropriate Coxeter–Dynkin diagram.

[2] There is a one-to-one correspondence between symbols used to decorate the Coxeter–Dynkin diagrams here and in previous papers. More precisely, $\bullet$, $\diamond$ and $\circ$ correspond, respectively, to a circle with a dot in the centre, an open square and a square with a cross in it in Moody & Patera (1992, 1993, 1995), and to a large open circle, an open square, and a square with a cross in it in Champagne *et al.* (1995).

In general, the number of copies of the face f_d, contained in a given polytope, is equal to the ratio of orders of the Coxeter groups (Champagne *et al.*, 1995),

$$\# f_d = \frac{|W(\mathfrak{g})|}{|G_s(f_d)|\ |G_f(f_d)|}, \tag{2}$$

where $|W|$ is the order of the corresponding reflection group. The subgroups G_s and G_f are read as sub-diagrams of the Coxeter–Dynkin diagram of $\mathfrak{g}$. Their nodes are identified by the appropriate decoration for f_d, namely $\circ$ for reflections generating G_s, and $\bullet$ for G_f.

The symmetry group of an intersection of two faces, say f_k and f_j of dimensions k and j, respectively, is the reflection group $G_f(f_k) \cap G_f(f_j)$ generated by reflections decorated by black circles and appearing in the diagram of either face. See the motivating example in §3.1.

Or, to answer the inverse of this question: given a face, say f_k, of a polytope, how many faces f_d of higher dimension, $k < d$, have f_k in common? Frequently, one would want to know the number of edges originating in the same vertex, $k = 0, d = 1$, or the number of faces f_d meeting in an edge ($k = 1$). The answer is the size of the orbit of the stabilizer $\mathrm{Stab}(f_k)$ in $W(\mathfrak{g})$ when it is acting on f_d. The stabilizer is the reflection group generated by the reflections labeled by circles of both colours in the decorated diagram of f_k.

2. Preliminaries

2.1. Finite reflection groups

The well known results of the classification of finite dimensional simple Lie algebras of any rank and type $n \geq 1$ are used here only to identify the finite reflection group $W(\mathfrak{g})$, called the Weyl group, or, equivalently, the crystallographic Coxeter group. In addition, we also consider polytopes with symmetries of finite non-crystallographic Coxeter groups denoted by H_2, H_3, H_4, together with their simple root diagrams.

The geometry of the set of simple roots (relative lengths and relative angles) in the real Euclidean space $\mathbb{R}^n$ is described by well known conventions implied in drawing the corresponding Coxeter–Dynkin diagrams (see, for example, Humphreys, 1990).

Reflections generating the Coxeter groups act in the n-dimensional real Euclidean space $\mathbb{R}^n$ spanned by the simple roots $\alpha_1, \ldots, \alpha_n$, according to

$$r_k x = x - \frac{2\langle \alpha_k, x\rangle}{\langle \alpha_k, \alpha_k\rangle}\alpha_k, \quad x \in \mathbb{R}^n, \quad k = 1, 2, \ldots, n, \tag{3}$$

that is, r_k is the reflection in the hyperplane of dimension $n-1$, containing the origin of $\mathbb{R}^n$, and orthogonal to the simple root α_k, $k \in \{1, \ldots, n\}$.

Instead of the α-basis of simple roots, we use the ω-basis. Two bases are linked by the Cartan matrix $\mathfrak{C}$ (for example, Humphreys, 1990; McKay & Patera, 1981)

Table 1
Number $|\Delta|$ of nonzero roots of simple Lie algebras $\mathfrak{g}$ and the orders $|W|$ of their Weyl group.

The number of roots and the orders of the three non-crystallographic Coxeter groups are shown.

$\mathfrak{g}$	A_n	B_n	C_n	F_4	H_2	H_3	H_4
$\lvert\Delta\rvert$	$n(n+1)$	$2n^2$	$2n^2$	48	10	120	120^2
$\lvert W\rvert$	$(n+1)!$	$n!\cdot 2^n$	$n!\cdot 2^n$	$2^7\cdot 3^2$	$2\cdot 5$	$2^3\cdot 3\cdot 5$	$2^6\cdot 3^2\cdot 5^2$

$$\alpha = \mathfrak{C}\omega, \mathfrak{C} = \left(\frac{2(\alpha_i|\alpha_j)}{(\alpha_j|\alpha_j)}\right), \quad i, j \in \{1, \ldots, n\}. \tag{4}$$

The decorations described in this paper should be viewed superimposed on the appropriate Coxeter–Dynkin diagram. Often, the same decoration applies to several diagrams. It should therefore be read in the context of that diagram when some specific information about the face of the polytope needs to be deduced from it.

2.2. Recursive diagram decoration rules

Decoration rules for connected Coxeter–Dynkin diagrams are recursive. They apply to diagrams of all simple Lie algebras and to any polytope generated by the corresponding Coxeter group starting from a single point in $\mathbb{R}^n$. They can therefore be written without reference to Platonic solids.

The 'grammar' for every decoration is as follows:

A node of the diagram can be decorated by $\diamond$, $\circ$, or by $\bullet$.

In a diagram of n-nodes, there can be up to n rhombi placed in any node.

Any connected pair of nodes must not carry $\circ$ and $\bullet$ side by side.

Assuming that a starting decoration complies with the grammar rules, the decoration describes a face of dimension equal to the number of $\bullet$ in it. For dual polytopes, the same decoration refers to the face of dimensions equal to the number of $\circ$. Usually one starts from vertices, faces of zero dimension, in which case there is no $\bullet$.

Decoration rules:

(i) Replace one of the rhombi by $\bullet$.

(ii) Change to $\diamond$ each $\circ$ that became adjacent to the new $\bullet$.

(iii) Repeat steps (i) and (ii) as long as there are any $\circ$.

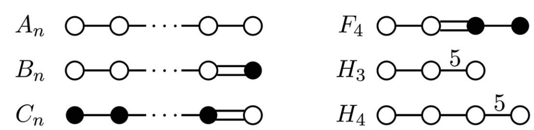
The following theorem is a direct consequence of the decoration rules.

Theorem 2.1. A polytope has one Coxeter group orbit of faces f_d for every dimension $0 \le d \le n-1$, *i.e.* it is Platonic, provided (i) the Coxeter–Dynkin diagram forms a connected line with no branches or loops, (ii) the seed decoration has one rhombus placed at either of the extreme nodes of the Coxeter–Dynkin diagram.

Proof. The Coxeter diagram describes all the finite symmetry groups in dimensions $n < \infty$.

The Platonic solids are polytopes with vertices generated by the action of only the Coxeter group starting from one vertex (seed point). Since Platonic solids have to have only a single orbit of faces of dimension $0 \le d < n$ the decoration rules must allow only one decoration for a diagram with d black circles. It is a straightforward consequence of the decoration rules that the diagram has to have nodes in one line and the seed point for the decoration has to have nonzero value attached to the extreme points. In all cases there are precisely two possibilities except for B_n and C_n, where out of four possibilities the interchange between long and short roots reduced to two possibilities. □

Corollary 2.1. Coxeter–Dynkin diagrams that give rise to Platonic solids in $\mathbb{R}^n$, where $n \ge 3$, are of the types



2.3. Dual polytopes

In this paper, we define dual polytopes and their faces using decorated Coxeter–Dynkin diagrams. The role of decoration elements $\circ$ and $\bullet$ are reversed in the dual polytope. That is, $\circ$ specifies reflections that generate the symmetry group G_f of the dual face, while $\bullet$ provides the generating reflections of the subgroup G_s that stabilizes the face pointwise.

The dual polytope of a Platonic polytope is also Platonic. Specifically, we have dual pairs of Platonic polytopes in $\mathbb{R}^n$, $n \ge 3$,

A_n: both polytopes coincide, but are differently oriented in $\mathbb{R}^n$;

B_n: B_n and C_n polytopes are dual to each other in $\mathbb{R}^n$;

F_4: the dual polytopes in $\mathbb{R}^4$ are different (see Table 4);

H_3: the dual polytopes are the icosahedron and dodecahedron in $\mathbb{R}^3$;

H_4: the dual polytopes are different in $\mathbb{R}^4$ (see Table 4).

A particular decoration that carries information about a face f_d of a polytope in $\mathbb{R}^n$ can be read as information about the face of the dual polytope, say $\tilde{f}_{n-d-1}$.

3. Platonic solids

3.1. Motivating example: Platonic solids in dimension 3

Let us illustrate the decoration method on the transparent example before presenting the general rules. Consider the classical Platonic solids in $\mathbb{R}^3$. Connected Coxeter–Dynkin diagrams of the groups, generated by three reflections, are of the types A_3, B_3, C_3 and H_3 (see Corollary 2.1).

We assume that diagram nodes are numbered 1, 2, 3 left to right, that each node represents the corresponding reflection r_1, r_2, r_3 acting in $\mathbb{R}^3$. Decorations described in this subsection should be viewed superimposed on any of the four diagrams (Corollary 2.1).

Let the seed point (face f_0) be either ω_1 or ω_3, corresponding to the dual pair of polytopes. We denote by $\diamond$ the reflections that move the seed point and by $\circ$ the reflections

Table 2
Properties of Platonic solids in $\mathbb{R}^3$.

Each line describes one face f_d in the Platonic solids, with either of the symmetry groups A_3, B_3, H_3. Reflections generating G_f and G_s are in the second and third columns. d contains the dimension of the face, $v = 2 - d$ shows the dimension of the corresponding face of the dual Platonic polytope. $N(A_3)$, $N(B_3)$ and $N(H_3)$ contain the number of times the face f_d occurs in the polytope. Vertically aligned marks in the last column indicate which faces belong to the same polytope of the dual pair.

Face	G_f	G_s	d	v	$N(A_3)$	$N(B_3)$	$N(H_3)$	Platonics	
◇○○	1	r_2, r_3	0	2	4	6	12	√	
○○◇	1	r_1, r_2	0	2	4	8	20		√
●◇○	r_1	r_3	1	1	6	12	30	√	
○◇●	r_3	r_1	1	1	6	12	30		√
●●◇	r_1, r_2	1	2	0	4	8	20	√	
◇●●	r_2, r_3	1	2	0	4	6	12		√

Table 3
Properties of the four-dimensional Platonic solids.

Underlying reflection groups in $\mathbb{R}^4$ are of the types A_4, B_4, F_4, H_4. For conventions, see the caption of Table 2.

Face	G_f	G_s	d	v	$N(A_4)$	$N(B_4)$	$N(F_4)$	$N(H_4)$	Platonics	
◇○○○	1	r_2, r_3, r_4	0	3	5	8	24	120	√	
○○○◇	1	r_1, r_2, r_3	0	3	5	16	24	600		√
●◇○○	r_1	r_3, r_4	1	2	10	24	96	720	√	
○○◇●	r_4	r_1, r_2	1	2	10	32	96	1200		√
●●◇○	r_1, r_2	r_4	2	1	10	32	96	1200	√	
○◇●●	r_3, r_4	r_1	2	1	10	24	96	720		√
●●●◇	r_1, r_2, r_3	1	3	0	5	16	24	600	√	
◇●●●	r_2, r_3, r_4	1	3	0	5	8	24	120		√

that stabilize it. Consider two seed-point decorations of all four diagrams (Corollary 2.1), namely

$$\diamond \circ \circ \qquad \text{seed point } \omega_1$$
$$\circ \circ \diamond \qquad \text{seed point } \omega_3.$$

Here, ◇ stands for the seed-point reflection. The sub-diagram decorated by ○○ identifies reflections generating G_s. The symmetry group $G_f(f_0) = 1$ of the face f_0 is trivial, the face is just a point.

The orders of reflection groups A_3, B_3, H_3 in $\mathbb{R}^3$ are shown in Table 1.

The groups G_s stabilizing ω_1 are, respectively,

$$|W(A_2)| = 6, \;\; |W(B_2)| = |W(C_2)| = 8, \;\; |W(H_2)| = 10. \quad (5)$$

Based on the number of vertices, we have a tetrahedron for A_3, an octahedron for B_3 and C_3, and a dodecahedron for H_3.

The groups G_s stabilizing ω_3 are all of type A_2. Hence, the numbers of vertices are those of a tetrahedron for A_3, a cube for B_3 and C_3, and an icosahedron for H_3. The dual tetrahedra of A_3 are differently oriented in $\mathbb{R}^3$, but otherwise are identical.

The second step describes the edges of the solids. The decorations are

$$\bullet \diamond \circ$$
$$\circ \diamond \bullet$$

The symmetry group generated by r_1 refers to the edge with end points ω_1 and $r_1\omega_1$. For the dual polytope, the symmetry group is generated by r_3, and it refers to the edge with end points ω_3 and $r_3\omega_3$. Note that this description of edges is independent of the Coxeter groups used in Corollary 2.1.

In all cases, the group $G_s(f_1) \times G_f(f_1)$ is $W(A_1 \times A_1)$ of order 4. Consequently, the number of edges is $|W(\mathfrak{g})|/4$. We get 6 for A_3 (tetrahedron), 12 for B_3 and C_3 (cube and octahedron) and 30 for H_3 (icosahedron and dodecahedron).

The final step in the decoration

$$\bullet \bullet \diamond \quad (6)$$

$$\diamond \bullet \bullet \quad (7)$$

describes the two-dimensional faces f_2. The sub-diagram ●● in equation (6) is the group $G_f(f_2) = W(A_2)$ of order $|W(A_2)| = 6$ for all cases.

Table 4
Summary of Proposition 3.1.

The first column contains the names of the four-dimensional Platonic polytopes and their duals; the second column shows the number of vertices in the polytope. The underlying symmetry group for each line of the table can be identified by comparing the number of vertices $\#f_0$ with the corresponding entries in Table 3. The third column contains the number $\#f_1(f_0)$ of edges meeting at each vertex, and the last column contains the number $\#f_2(f_1)$ of two-dimensional faces meeting at each edge.

Name of polytope	$\#f_0$	$\#f_1(f_0)$	$\#f_2(f_1)$
Pentatope	5	4	3
16-cell	8	6	4
Tessaract	16	4	3
24-cell	24	8	3
600-cell	120	20	5
120-cell	600	4	3

Table 5
Properties of Platonic solids of diemsions ≥ 5.

Underlying reflection groups in $\mathbb{R}^n$ ($n \geq 5$) are of the types A_n, B_n.

Face	G_f	G_s	d	v	$N(A_n)$	$N(B_n)$	Platonics	
◇○○...○○	1	$r_2, \ldots, r_n$	0	$n-1$	$(n+1)!/n!$	$2^n n!/2^{n-1}(n-1)!$	✓	
○○○...○◇	1	$r_1, \ldots, r_{n-1}$	0	$n-1$	$(n+1)!/n!$	$2^n n!/n!$		✓
●◇○...○○	r_1	$r_3, \ldots, r_n$	1	$n-2$	$(n+1)!/2!(n-1)!$	$2^n n!/2!2^{n-2}(n-2)!$	✓	
○○○...◇●	r_n	$r_1, \ldots, r_{n-2}$	1	$n-2$	$(n+1)!/(n-1)!2!$	$2^n n!/(n-1)!2!$		✓
●●◇...○○	r_1, r_2	$r_4, \ldots, r_n$	2	$n-3$	$(n+1)!/3!(n-2)!$	$2^n n!/3!2^{n-3}(n-3)!$	✓	
○○◇...●●	r_{n-1}, r_n	$r_1, \ldots, r_{n-3}$	2	$n-3$	$(n+1)!/(n-2)!3!$	$2^n n!/(n-2)!2^2 2!$		✓
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
●●●...◇○	$r_1, \ldots, r_{n-2}$	r_n	$n-2$	1	$(n+1)!/(n-1)!2!$	$2^n n!/2!(n-1)!$	✓	
○◇●...●●	$r_3, \ldots, r_n$	r_1	$n-2$	1	$(n+1)!/2!(n-1)!$	$2^n n!/2!2^{n-2}(n-2)!$		✓
●●●...●◇	$r_1, , r_{n-1}$	1	$n-1$	0	$(n+1)!/n!$	$2^n n!/n!$	✓	
◇●●...●●	$r_2, \ldots, r_n$	1	$n-1$	0	$(n+1)!/n!$	$2^n n!/2^{n-1}(n-1)!$		✓

$$\#f_2 : \frac{|W(A_3)|}{|W(A_2)|} = 4, \;\; \frac{|W(B_3)|}{|W(A_2)|} = \frac{|W(C_3)|}{|W(A_2)|} = 8, \;\; \frac{|W(H_3)|}{|W(A_2|)} = 20.$$

The same sub-diagram in equation (7) stands for $G_f(f_2) = W(A_2)$ in the case of A_3, $G_f(f_2) = W(C_2)$ for B_3 and C_3, and $G_f(f_2) = W(H_2)$ for H_3. Therefore

$$\#\tilde{f}_2 : \frac{|W(A_3)|}{|W(A_2)|} = 4, \;\; \frac{|W(B_3)|}{|W(C_2)|} = \frac{|W(C_3)|}{|W(C_2)|} = 6, \;\; \frac{|W(H_3)|}{|W(H_2|)} = 12.$$

The shape of face f_2 can be easily determined from the relative angles of the mirrors r_1 and r_2 acting on ω_1 or from reflections r_2 and r_3 acting on ω_3.

A summary of properties of Platonic solids in $\mathbb{R}^3$ and their faces is shown in Table 2.

Example 1. The dual pair of Platonic solids of A_3 consists of two tetrahedra differently oriented in $\mathbb{R}^3$, their seed points being ω_1 and ω_3.

The vertices of the two tetrahedra are as follows:

$$\omega_1, -\omega_1 + \omega_2, -\omega_2 + \omega_3, -\omega_3$$
$$\omega_3, -\omega_3 + \omega_2, -\omega_2 + \omega_1, -\omega_1.$$

3.2. Platonic solids in dimension 4

Connected Coxeter–Dynkin diagrams (nodes are connected in one line) of the groups generated by four reflections are of the types A_4, B_4, C_4, F_4 and H_4 (see Corollary 2.1). One may notice that the diagram of type D_4 is excluded.

Using Table 2 conventions, properties of the four-dimensional Platonic solids, together with their duals, are summarized in Table 3.

Let us answer the question posed in §1: how many edges f_1 meet in a vertex f_0 of any Platonic solid?

Decorations of Coxeter–Dynkin diagrams for four-dimensional Platonic solids and their duals are:

$$f_0 : \diamond \circ \circ \circ \qquad f_1 : \bullet \diamond \circ \circ \tag{8}$$

$$\tilde{f}_0 : \circ \circ \circ \diamond \qquad \tilde{f}_1 : \circ \circ \diamond \bullet \tag{9}$$

From Table 3, we read the stabilizers of faces

$$G_s(f_0) = \langle r_2, r_3, r_4 \rangle \qquad G_s(f_1) = \langle r_3, r_4 \rangle \tag{10}$$

$$G_s(\tilde{f}_0) = \langle r_1, r_2, r_3 \rangle \qquad G_s(\tilde{f}_1) = \langle r_1, r_2 \rangle. \tag{11}$$

The edges originating in f_0 or $\tilde{f}_0$ are generated by the stabilizer of f_0 or $\tilde{f}_0$, respectively.

They are equal to $G_s(f_1)$ for equation (10) and $G_s(\tilde{f}_1)$ for equation (11).

The formula for the number of edges meeting in one vertex can be written as

$$\#f_1(f_0) = \frac{|G_s(f_0)|}{|G_s(f_1)|}, \tag{12}$$

or more generally, if we consider $f_d(f_{d-1})$ of faces f_d having in common faces f_{d-1} for $1 \leq d \leq n-2$:

Proposition 3.1. The number of faces f_d meeting in a face f_{d-1} for $d \in \{1, \ldots n-2\}$ is equal to the ratio of orders of Coxeter subgroups $G_s(f_d)$, $G_s(f_{d-1})$, which stabilize a given face f_d, f_{d-1}, respectively,

$$\#f_d(f_{d-1}) = \frac{|G_s(f_{d-1})|}{|G_s(f_d)|}. \tag{13}$$

The groups $G_s(f_1)$ for equation (8) are, respectively, $W(A_2)$, $W(B_2)$, $W(A_2)$ and $W(H_2)$, and for equation (9) $G_s(\tilde{f}_1) = W(A_2)$.

Table 4 summarizes this proposition.

Both the pentatope and 24-cell are self-dual, the 16-cell is dual to the tessaract, and the 600-cell and 120-cell are dual to each other (Coxeter, 1973).

3.3. Platonic solids in dimension ≥5

There are only three types of Platonic solids for dimension more than 4, namely the simplex, hypercube and cross-polytope. The simplex is self-dual, and the cross-polytope and hypercube are dual to each other. The corresponding Coxeter–Dynkin diagrams (Corollary 2.1) are of the types A_n, B_n and C_n.

Using Table 2 conventions, the properties of Platonic solids of dimension ≥ 5, together with their duals, are summarized in Table 5.

Table 6 summarizes this Proposition 3.1 for any n-dimensional polytope.

Table 6
Summary of Proposition 3.1 for any n-dimensional polytope.

The first column contains the names of the n-dimensional Platonic polytopes and their duals; the second column shows the number of vertices in the polytope; the remaining columns contain the number of faces f_d meeting at each face f_{d-1} for $d \in \{1, \ldots, n-2\}$, $n \geq 5$

Name of polytope	$\#f_0$	$\#f_1(f_0)$	$\#f_2(f_1)$	$\ldots$	$\#f_{n-3}(f_{n-4})$	$\#f_{n-2}(f_{n-3})$
Simplex [$(n+1)$-cell]	$n+1$	n	$n-1$	$\ldots$	4	3
Cross-polytope (2^n-cell)	$2n$	$2(n-1)$	$2(n-2)$	$\ldots$	8	4
Hypercube ($2n$-cell)	2^n	n	$n-1$	$\ldots$	4	3

Analogously to Proposition 3.1, we can give the formula for finding the number of faces f_d meeting at face f_c, where $c < d$.

Proposition 3.2. The number of faces f_d meeting at face f_c for $0 \leq c < d < n-2$ is equal to the ratio of orders of Coxeter subgroups $G_s(f_c)$, $G_s(f_d)$, which stabilize a given face f_c and f_d, respectively,

$$\#f_d(f_c) = \frac{|G_s(f_c)|}{|G_s(f_d)|}. \tag{14}$$

We now consider the four-dimensional faces appearing in five-dimensional polytopes.

Diagrams of the face f_4 are:

●●●●◇

◇●●●●

The shape of face f_4 is determined from the relative angles of mirrors $r_1, \ldots, r_4$ acting on ω_1, and from reflections $r_2, \ldots, r_5$ acting on ω_5. Using Tables 3 and 4, one can see from the subdiagram ●●●● that, for A_5, faces are pentatopes, and for B_5, either 16-cells or tessaracts.

4. Concluding remarks

A seed decoration is often applied to several disconnected Coxeter diagrams. See an example in §3.2. The same is true for recursive decorations that follow from the seed. Therefore, the set of such decorations could be viewed as a 'generic polytope' pertinent to all Coxeter groups with the same connectivity of their Coxeter–Dynkin diagrams. Can anything be learned from the generic set of decorations?

In this paper, we focused on Coxeter groups whose diagrams are connected. In general, the decoration rules can be used for polytopes of Lie algebras that are semi-simple but not simple, *i.e.* their Coxeter–Dynkin diagrams are disconnected. In this case, each connected component must have its own seed point, indicated in the initial decoration. Consequently, there are several orbits of edges, and hence such polytopes are never Platonic.

The diagram decoration method used here can also be applied for other polytopes generated by finite reflection groups, for example the Archimedean polytopes (see table in Champagne *et al.*, 1995) and the root polytopes (Cellini & Marietti, 2014; Mészáros, 2011).

In Champagne *et al.* (1995), the authors considered semiregular polytopes in three and four dimensions. It would be interesting to describe semiregular polytopes in a higher dimension.

In recent years evidence has been obtained that there exist in nature molecules with imperfect symmetries (Bodner *et al.*, 2013, 2014; Fowler & Manolopoulos, 2007; McKenzie *et al.*, 1992) of the $W(\mathfrak{g})$ type, not necessarily Platonic. They could be considered as $W(\mathfrak{g})$ symmetries broken to the symmetry of a subgroup.

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